Newtonian Labs Teaching Guides

Physics of the Magneto-Mechanical Harmonic Oscillator

Table of Contents

1 Introduction

The Harmonic Oscillator (sometimes called the Simple Harmonic Oscillator, or SHO) plays a central role in modern physics and technology. For example, the mathematics of harmonic motion can be used to describe the behavior of mechanical systems, electromagnetic systems, quantum mechanical systems, acoustic systems, and a broad range of other physical phenomena. Moreover, this same mathematics provides the foundation for the development of many fundamental topics in theoretical physics, including wave mechanics, quantum mechanics, and quantum field theory.

The simple *mechanical* harmonic oscillator is a topic going back hundreds of years, yet even here modern incarnations abound. The realization of microfabricated high-Q mechanical oscillators showing quantum behavior is currently a very hot topic, with many scientific groups exploring this technology and its potential applications. High-Q mechanical oscillators made from small plates of quartz crystal also form the foundation of timekeeping and frequency reference in electronic devices. Essentially very cell phone, timepiece, microwave transmitter, computer, and many other electronic devices contain quartz crystal oscillators, and they are currently being manufactured at a

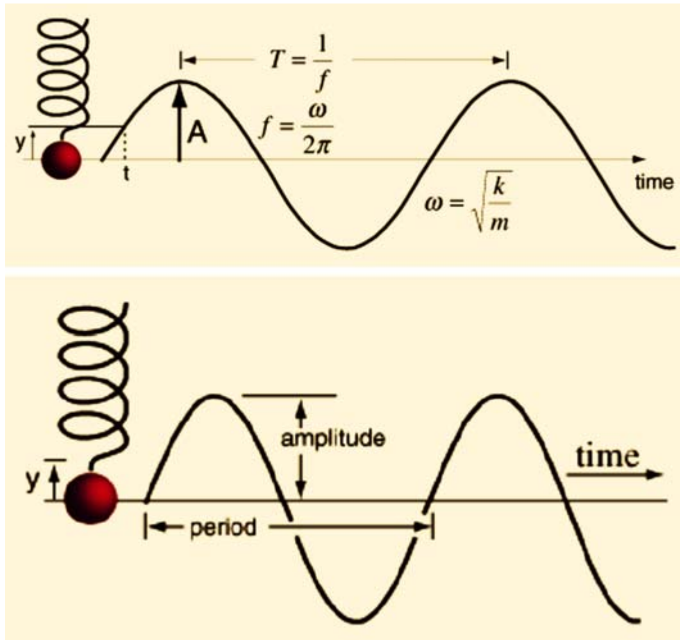

Figure 1. Basic terminology and notation for simple harmonic motion. (Image source: http://hyperphysics.phy-astr.gsu.edu/hbase/shm.html)

rate of about two *billion* units per year.

The focus of the Magneto-Mechanical Harmonic Oscillator (MMHO) experiment is on understanding the Harmonic Oscillator, using a large-scale example where one can see rather directly how it responds to various stimuli.

2 The Simple Harmonic Oscillator

Before we look at the specific apparatus, let us first review the mathematics of simple harmonic motion, thereby defining our variables and examining how oscillators behave. We begin with the canonical example of a mass on a spring, as shown in Figure 1. One thing that makes this a *simple* harmonic oscillator is that we assume a purely linear, Hooke's-law spring constant, giving a restoring force $F = -kx$, where k is a constant. We also neglect the mass of the spring, so the load mass m is a simple constant. In the absence of any damping, the equation of motion for this system is then

$$\begin{aligned} F &= ma = -kx \\ m\frac{d^2x}{dt^2} &= m\ddot{x} = -kx \\ m\ddot{x} + kx &= 0 \end{aligned}$$

Solutions to this equation are of the form

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \tag{1}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2}$$

while C_1 and C_2 are arbitrary constants. The constant $\nu_0 = \omega_0/2\pi$ is called the *resonant frequency* of the oscillator, measured in Hertz. (The constant ω_0 is also sometimes called the resonant frequency, although more precisely ω_0 is the resonant *angular frequency*, this being measured in radians per second. Frequency measurements are usually reported in Hertz, while ω_0 is often more convenient for doing theory.) Note that the equation of motion is often written in the convenient form

$$\ddot{x} + \omega_0^2 x = 0 \tag{3}$$

The general theory of differential equations (not covered here!) tells us that Equation 1 is the full and unique solution to this equation of motion. All we need to supply is the appropriate choice of C_1 and C_2. For example, if we know the initial position $x(t=0)$ and velocity $\dot{x}(t=0)$, then plugging in these *initial conditions* allows us to solve for C_1 and C_2, and from this we can predict the motion $x(t)$ for all future times.

We often use complex notation when talking about harmonic oscillators, for reasons described in the Appendix below. The (complex) solution to Equation 3 is $x(t) = \tilde{A}e^{i\omega_0 t}$, where $\tilde{A}$ is a complex constant, as the reader can quickly verify. The physical solutions are then either the real or imaginary parts of the complex solution, giving two arbitrary constants (the real and imaginary parts of $\tilde{A}$), and these constants are related to the C_1 and C_2 in Equation 1. The complex notation is often exceedingly useful because of its simplicity (dealing with $e^{i\omega t}$ is less cumbersome compared with sines and cosines), and we will see this more below. But one should note that this is a bit of a shorthand notation, which can lead to problems. In the final analysis it is the real, physical, solutions that describe the system.

2.1 Energetics

Consider the general oscillator solution in Equation 1, which we can also write in the form $x(t) = A\sin(\omega_0 t + \delta)$, where A and δ are real constants. The kinetic energy of the mass motion is

$$\begin{aligned} E_{kinetic} &= \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \\ &= \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t + \delta) \end{aligned}$$

and the potential energy stored in the spring is

$$\begin{aligned} E_{potential} &= \int_0^x F(x')dx' = \int_0^x kx'dx' \\ &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}kA^2 \sin^2(\omega_0 t + \delta) \\ &= \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \delta) \end{aligned}$$

where we used Equation 2 to obtain the last expression.

From these we see that the total energy

$$\begin{aligned} E_{total} &= E_{kinetic} + E_{potential} \qquad (4) \\ &= \frac{1}{2}m\omega_0^2 A^2 \end{aligned}$$

is independent of time. As the mass oscillates, energy sloshes back and forth between kinetic energy and potential energy.

3 The Damped Harmonic Oscillator

We add damping to our simple harmonic oscillator using the damping force $F = -\gamma\dot{x}$, where γ is a constant. (It is possible that γ is not constant in some physical systems; in this case the system cannot be described as a *simple* harmonic oscillator.) The equation of motion $(F = ma)$ then becomes

$$\begin{aligned} m\ddot{x} &= -kx - \gamma\dot{x} \qquad (5) \\ m\ddot{x} + \gamma\dot{x} + kx &= 0 \end{aligned}$$

which is often written

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = 0 \tag{6}$$

where $\Gamma = \gamma/m$ and again we use $\omega_0^2 = k/m$.

To solve Equation 6, we try a solution of the form $x = \tilde{A}e^{i\omega t}$, where $\tilde{A}$ is a complex constant and ω is a real constant. Plugging this in gives

$$\left(-\omega^2 + i\omega\Gamma + \omega_0^2\right)x = 0$$

Assuming $x \neq 0$, we have

$$\omega^2 - i\omega\Gamma - \omega_0^2 = 0$$

and solving this quadratic equation gives

$$\omega = \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}} \tag{7}$$

So our final real solution is $x = \mathrm{Re}[\tilde{A}e^{i\omega t}]$, where $\tilde{A}$ is a complex constant and ω is given by Equation 7.

At this point we will limit ourselves to the regime $\Gamma^2 < 4\omega_0^2$, which is called the *underdamped* case. In this case the test mass oscillates away, as in the zero damping case, but the oscillations slowly decrease in amplitude with time, eventually settling to $x = 0$. The opposite limit, $\Gamma^2 > 4\omega_0^2$, is called the *overdamped* case, and here the test mass simply damps down without oscillating, like a pendulum in molasses. The cross-over, $\Gamma^2 = 4\omega_0^2$, is called the *critically damped* case. Most modern science and technology applications focus on underdamped oscillators, and the MMHO was designed to explore this regime in depth.

To simplify the notation a bit, we define the damped oscillator frequency ω_d, where

$$\omega_d^2 = \omega_0^2 - \frac{\Gamma^2}{4}$$

and we further define the decay time T

$$T = \frac{2}{\Gamma}$$

so we can write the full (still complex) solution

$$x(t) = \tilde{A}e^{-t/T}e^{\pm i\omega_d t}$$

where $\tilde{A}$ is a complex constant while T and ω_d are both real numbers. Converting to only real quantities, the full solution becomes

$$x(t) = C_1 e^{-t/T}\cos(\omega_d t) + C_2 e^{-t/T}\sin(\omega_d t) \tag{8}$$

where C_1 and C_2 are real constants. And again, these constants are determined by the initial conditions of our oscillator – the position and velocity at $t = 0$. This solution can also be written

$$x(t) = C_3 e^{-t/T}\cos(\omega_d t + \delta)$$

where now C_3 and δ are real constants. One can expand $\cos(\omega_d t + \delta)$ to find the relationship between (C_1, C_2) and (C_3, δ), so either set of constants can be derived from the other.

Broadly speaking, the underdamped oscillator looks a lot like the undamped oscillator, except that the amplitude of the oscillations decays exponentially with a time constant T.

3.1 The Quality Factor

We also define a *quality factor* Q for the oscillator,

$$\begin{aligned} Q &= \pi\frac{\text{Decay Time}}{\text{Resonant Period}} \\ &= \pi\frac{T}{2\pi/\omega_d} \\ Q &= \frac{\omega_d T}{2} = \frac{\omega_d}{\Gamma} \end{aligned}$$

Note that Q is a dimensionless number, roughly equal to the number oscillation cycles that occur before the amplitude decays away. (More precisely, the amplitude decays to $e^{-\pi}$ times its original value after Q cycles.) This is often written as

$$Q = 2\pi \frac{\text{Energy Stored}}{\text{Energy Loss per Cycle}}$$

and the reader can verify that these two definitions are the same.

The MMHO is a damped harmonic oscillator with $Q > 100$. In this case we can expand ω_d for small Q^{-1}, giving

$$\begin{aligned}
\omega_d &= \left(\omega_0^2 - \frac{\Gamma^2}{4}\right)^{1/2} \\
&= \omega_0 \left(1 - \frac{\Gamma^2}{4\omega_0^2}\right)^{1/2} \\
&\approx \omega_0 \left(1 - \frac{1}{2Q^2}\right)
\end{aligned}$$

What this means is that ω_d will equal ω_0 to better than a part in 10^4, and such a small difference will be negligible in this experiment. We will therefore assume $\omega_d = \omega_0$ in the discussion that follows, and this greatly simplifies the math. The quality factor Q can then be written

$$\begin{aligned}
Q &= \frac{\omega_0 T}{2} = \frac{\omega_0}{\Gamma} \\
&= \frac{2\pi\nu_0}{\Gamma} = \pi\nu_0 T
\end{aligned}$$

Note that the quality factor Q and the damping constant Γ are related. For an underdamped oscillator, we can talk about the damping constant Γ, or the decay time T, or the quality factor Q. They all refer to basically the same thing, that the oscillation amplitude decays away with time. A good tuning fork might have a Q of several thousand, and mechanical oscillators with $Q > 10^6$ are not uncommon these days in high-tech applications.

4 The Driven Harmonic Oscillator

Finally, we take the last step and drive our oscillator with a sinusoidal force $F(t) = F_{applied} \cos \omega t$, giving the equation of motion

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = (F_{applied}/m) \cos \omega t \tag{9}$$

The force is often written as a complex expression $F(t) = F_{applied} e^{i\omega t}$ (where $F_{applied}$ is still a real number), giving the complex equation of motion

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = (F_{applied}/m)\, e^{i\omega t} = F_1 e^{i\omega t} \tag{10}$$

where we have defined a normalized force $F_1 = F_{applied}/m$.

To solve this equation, we try a solution of the form $x(t) = \tilde{B}e^{i\omega t}$, where $\tilde{B}$ is a complex constant, and plugging this in gives

$$\begin{aligned}
\left(-\omega^2 + i\omega\Gamma + \omega_0^2\right) \tilde{B}e^{i\omega t} &= F_1 e^{i\omega t} \qquad (11) \\
\tilde{B} &= \frac{F_1}{(\omega_0^2 - \omega^2) + i\omega\Gamma}
\end{aligned}$$

4.1 The Full Solution

The solution shown in Equation 11 is called a *particular* solution to the equation of motion; it is not the full solution. The full, real solution to Equation 10 can be written:

$$x(t) = \mathrm{Re}\left[\tilde{A}e^{-t/T} e^{i\omega_d t} + \frac{F_1}{(\omega_0^2 - \omega^2) + i\omega\Gamma} e^{i\omega t}\right]$$

where $\tilde{A}$ is a complex constant that depends on the initial conditions. We will not prove here that this is the full, unique solution; that is a task for a course in differential equations. For now we just note that this $x(t)$ does satisfy Equation 10, and it contains two real constants (the real and imaginary parts of $\tilde{A}$) that we adjust to satisfy the initial conditions.

4.2 The Steady-State Response Function

The full solution is complicated, but you can see that for $t \gg T$ the first term goes to zero. When $t \gg T$ we are left with the *steady-state solution*

$$x(t) = \mathrm{Re}\left[\frac{F_1}{(\omega_0^2 - \omega^2) + i\omega\Gamma} e^{i\omega t}\right] \tag{12}$$

Note that in steady-state $x(t)$ oscillates at the drive frequency ω (as seen by the $e^{i\omega t}$ term), which is generally *not* equal to the resonant frequency ω_0. Note also that the steady-state solution is independent of $\tilde{A}$, which means that in steady-state the motion of the oscillator no longer depends on the initial conditions.

To summarize, if we drive our oscillator with an applied force $F(t) = F_{applied}\cos\omega t$, the resulting motion of the oscillator can be written $x(t) = A\cos(\omega t + \delta)$, where A and δ are real constants and

$$A = \left|\frac{F_1}{(\omega_0^2 - \omega^2) + i\omega\Gamma}\right|$$

You can see immediately from this expression that if you drive the oscillator near its resonant frequency $(\omega \approx \omega_0)$, then the oscillation amplitude will be high. If you drive it far away from resonance, the amplitude will be lower.

Expanding this expression gives

$$\begin{aligned}
A &= \sqrt{\frac{F_1}{(\omega_0^2 - \omega^2) + i\omega\Gamma} \cdot \frac{F_1}{(\omega_0^2 - \omega^2) - i\omega\Gamma}} \\
\frac{A}{F_1} &= \frac{1}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\omega\Gamma\right)^2}} \\
4\pi^2\frac{A}{F_1} &= \frac{1}{\sqrt{\left(\nu^2 - \nu_0^2\right)^2 + \left(\nu\Gamma/2\pi\right)^2}}
\end{aligned}$$

This function is sometimes called the *response function* of the oscillator. It gives the amplitude of the oscillations as a function of the drive frequency.

4.3 Steady-State Behavior Near Resonance

When the drive frequency is near resonance, we can write $\omega = \omega_0 + \Delta\omega$ (or $\nu = \nu_0 + \Delta\nu$) and expand for small $\Delta\omega$, giving

$$\begin{aligned}
\omega_0^2 - \omega^2 &= (\omega_0 + \omega)(\omega_0 - \omega) \\
&\approx -2\omega_0\Delta\omega
\end{aligned}$$

and the response function becomes

$$\begin{aligned}
\frac{A}{F_1} &\approx \frac{1}{2\omega_0}\frac{1}{\sqrt{\left(\Delta\omega\right)^2 + \Gamma^2/4}} \qquad (13) \\
4\pi^2\frac{A}{F_1} &\approx \frac{1}{2\nu_0}\frac{1}{\sqrt{\left(\Delta\nu\right)^2 + \left(\Gamma/4\pi\right)^2}}
\end{aligned}$$

which is a Lorentzian function.

Note that the amplitude peaks when the oscillator is driven at its resonant frequency $(\omega = \omega_0, \Delta\omega = 0)$, giving

$$A_{\text{on resonance}} = \frac{F_1}{\omega_0\Gamma} = \frac{Q}{\omega_0^2}F_1 = \frac{Q}{m\omega_0^2}F_{applied}$$

From this we see that the amplitude of a driven oscillator is proportional to Q when driven on resonance. With a very high quality factor, only a small driving force is needed to produce a large oscillation amplitude.

4.4 Steady-State Behavior Far From Resonance

If we drive the oscillator at frequencies far below resonance, the amplitude of the oscillator becomes

$$A_{\text{static}} \approx \frac{1}{\omega_0^2}F_1 = \frac{1}{m\omega_0^2}F_{applied}$$

This low-frequency limit essentially gives us the static response of the oscillator. We can also get this by going all the way back to the beginning of our discussion. Hooke's law gives a restoring force $-kx$. Setting this equal to the applied force gives $F_{applied} = kx_{static}$, so $x_{static} = F_{applied}/k = F_{applied}/m\omega_0^2$.

If we drive the oscillator at frequencies far above resonance, the response becomes

$$A_{\text{highfrequency}} \approx -\frac{F_1}{\omega^2} = -\frac{F_{applied}}{m\omega^2} \tag{14}$$

This is exactly what happens when you apply a sinusoidal force to a free mass. Newton's law for a free mass is simply (with no restoring force and no damping)

$$m\ddot{x} = F_{applied}\cos(\omega t)$$

Assuming a solution $x(t) = A\cos(\omega t)$ gives

$$\begin{aligned} -mA\omega^2\cos(\omega t) &= F_{applied}\cos(\omega t) \qquad (15) \\ A_{\text{highfrequency}} &= -\frac{F_{applied}}{m\omega^2} \end{aligned}$$

4.5 Phase Information

It is also instructive to look at the steady-state phase of the driven oscillator relative to the phase of the applied force $F(t)$. At low frequencies, we saw in the last section that x_{static} is simply proportional to $F_{applied}$, meaning that the oscillator displacement is in phase with the drive. On the other hand, far above the resonance frequency, Equation 15 tells us that x is 180 degrees out of phase with the drive. When we are pushing to the left, the position of the mass is to the right, and vice versa.

When the applied force is on resonance, we have to go back to the complex amplitude in Equation 11, which becomes

$$\tilde{B} = \frac{F_1}{i\omega\Gamma}$$

and this tells us that the oscillator position is 90 degrees out of phase with the drive. Thus we see the transition – from in-phase at low drive frequencies (the static response), to 90 degrees out of phase on resonance, to 180 degrees out of phase at high drive frequencies (the free-mass response).

5 The Torsional Oscillator

The MMHO is a *torsional oscillator*, which is a special form of a simple harmonic oscillator. In a torsional oscillator, linear motion is replaced by angular motion. So the displacement $x(t)$ is replaced by the angular displacement $\theta(t)$. Newton's law $F = ma$ is replaced by its torque version

$$\tau = I\ddot{\theta}$$

where $\ddot{\theta} = d^2\theta/dt^2$ is the angular acceleration, and I is the mass moment of inertia The restoring force becomes a restoring torque

$$\tau = -\kappa\theta$$

The math all follows exactly the same as it did above, giving us a resonance frequency ω_0, a decay time T, a quality factor Q, etc. The resonant frequency in the torsional case, for example, becomes

$$\omega_0 = \sqrt{\frac{\kappa}{I}} \tag{16}$$

If we drive the oscillator with a sinusoidal torque, $\tau(t) = \tau_{applied} \cos \omega t$, then the steady-state response can be written $\theta(t) = A\cos(\omega t + \delta)$ and we again obtain the response function described above.

6 Magnetic Drive

Since the test mass in the MMHO includes a permanent magnet, we can drive the torsional oscillator magnetically. If we create a magnetic field $\vec{B}(t)$ at the position of the test mass, then the torque on the test mass is

$$\tau_{magnetic}(t) = \vec{\mu} \times \vec{B}(t)$$

where $\vec{\mu}$ is the magnetic moment of the magnet. With a sinusoidal current applied to the drive coil, this becomes

$$\tau_{magnetic}(t) = \mu_0 B_0 \cos\theta \cos(\omega t)$$

where μ_0 is the size of the magnetic moment and B_0 is the amplitude of the oscillating magnetic field. For small oscillation amplitudes, we can use the small-angle approximation $\cos\theta \approx 1$, giving a simple sinusoidal driving torque

$$\tau_{magnetic}(t) = \mu_0 B_0 \cos(\omega t)$$

7 Adding a Magnetic Restoring Force

In the MMHO it is also possible to apply a magnetic restoring torque, adding this to the restoring torque from the support wires. If we apply a constant magnetic field of strength B_0 in the $\theta = 0$ direction (from the bias coils), then the magnetic torque on the test mass is

$$\begin{aligned} \tau_{magnetic} &= \vec{\mu} \times \vec{B} \\ &= -\mu_0 B_0 \sin\theta \end{aligned}$$

where θ is the angular position of the test mass. Using the small-angle approximation $\sin\theta \approx \theta$, we can write the total restoring torque

$$\begin{aligned} \tau &= -\kappa_0\theta - \mu_0 B_0 \theta \\ &= -(\kappa_0 + \mu_0 B_0)\theta \\ &= -\kappa_{total}\theta \end{aligned}$$

where μ_0 is the magnetic moment of the test mass and κ_0 is the spring constant provided by the support wires. With a nonzero B_0, the resonant frequency of the oscillator becomes

$$\omega_0 = \sqrt{\frac{\kappa_{total}}{I}}$$

and for small B_0 the frequency change is

$$\begin{aligned} \Delta\omega_0 &\approx \frac{1}{2}\left(\frac{\kappa_{total}}{I}\right)^{-1/2} \Delta\kappa_{total} \\ \frac{\Delta\omega_0}{\omega_0} &\approx \frac{1}{2}\frac{\Delta\kappa_{total}}{\kappa_{total}} \\ &\approx \frac{1}{2}\frac{\mu_0 B_0}{\kappa_0} \end{aligned}$$

This change in the resonant frequency of the oscillator allows you to measure the magnetic moment of the test mass. You first calculate I, then combine this with the known resonant frequency ω_0 to determine κ_0. Then you measure $\Delta\omega_0/\omega_0$ as a function of the applied B_0 to determine μ_0.

8 Appendix: Using Complex Functions to Solve Real Equations

Physicists and engineers often use complex functions to solve real equations, with the understanding that you take the real part at the end. Why does this work? And why do we even do this? We can demonstrate with the simple harmonic oscillator. Start with the equation of motion $\ddot{x} + \omega_0^2 x = 0$, and let us solve this using a complex function: $x = \alpha + i\beta$, where $\alpha(t)$ and $\beta(t)$ are real functions. If you plug this in, you will see that $\ddot{x} + \omega_0^2 x = 0$ becomes

$$\begin{aligned} \left(\ddot{\alpha} + i\ddot{\beta}\right) + \omega_0^2 \left(\alpha + i\beta\right) &= 0 \\ \left(\ddot{\alpha} + \omega_0^2 \alpha\right) + i\left(\ddot{\beta} + \omega_0^2 \beta\right) &= 0 \end{aligned} \tag{17}$$

Since a complex number equals zero only if both the real and imaginary parts equal zero, we see that $\ddot{x} + \omega_0^2 x = 0$ implies that both $\ddot{\alpha} + \omega_0^2 \alpha = 0$ and $\ddot{\beta} + \omega_0^2 \beta = 0$. In other words, both the real and imaginary parts of $x(t)$ satisfy the original equation.

So we have a procedure: try using a complex function to solve the original equation. If this works, then taking the real part of the solution gives a real function that also solves the same differential equation. (If in doubt, then verify directly that the real part solves the equation.)

Why do we go to the trouble of using complex functions to solve a real equation? Because differential equations are often easier to solve when we assume complex functions (seems counterintuitive, but it's true). The function $e^{i\omega t}$ is a simple exponential, and the derivative of an exponential is another exponential – that makes things simple. In contrast, cosines and sines are more difficult to work with.

In the case of the simple harmonic oscillator, the solution $x(t) = \tilde{A}e^{i\omega t}$, where $\tilde{A}$ is complex, has a natural interpretation. The length and angle of the $\tilde{A}$ vector (in the complex plane) give the amplitude and phase of the oscillations.

You should note, however, that this only works for linear equations. If our equation were $\ddot{x} + \omega_0^2 x + \gamma x^2 = 0$, for example, then using complex functions would not have the same benefits. In fact, there is no simple solution to this equation, complex or otherwise. This equation describes a nonlinear oscillator, and nonlinear oscillators exhibit a fascinating dynamics with interesting behaviors that people still study to this day.